Conservation laws for second-order invariant variational problems

1984 J. Phys. A: Math. Gen. 17 3425

(http://iopscience.iop.org/0305-4470/17/18/012)

J. Phys. A: Math. Gen. 17 (1984) 3425-3428. Printed in Great Britain

Conservation laws for second-order invariant variational problems

J David Logan† and John B Bdzil‡

† Department of Mathematics and Statistics, University of Nebraska, Lincoln NE 68588, USA.

‡ Group M-3, Los Alamos National Laboratory, Los Alamos NM 87545, USA.

Received 29 June 1984

Abstract. In the literature, a symmetry requirement in the mixed, second partials appearing in the Lagrangian is needed before conservation laws for second-order variational problems can be obtained. In this paper this assumption is removed and an example of a modified, linear Korteweg-de Vries equation is analysed.

1. Introduction

In studying compression waves in non-homogeneous media, it was recently noticed by one of the authors (JBB) that the second-order variational functional

$$J(u(x, t)) = \iint (\tfrac{1}{2}\delta^2 u_t u_x + \tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_{tx}^2)\, \mathrm{d}t\, \mathrm{d}x, \tag{1.1}$$

whose Euler equation is a modified, linear Korteweg–de Vries equation

$$u_{tt} + \delta^2 u_{tx} + u_{ttxx} = 0, \tag{1.2}$$

did not lead to a conservation law (due to time invariance) as predicted by the theory presented in the papers by Blakeslee and Logan (1976, 1977), Logan and Blakeslee (1975) and in the monograph by Logan (1977). Rather, it was necessary to symmetrise the Lagrangian in the term involving the mixed second partial by replacing u_{tx}^2 by $\frac{1}{2}u_{tx}^2 + \frac{1}{2}u_{xt}^2$ before a conservation law could be obtained. On careful examination it was found that the results in the papers by Blakeslee and Logan depend upon the assumption that

$$\partial L/\partial u_{xt} = \partial L/\partial u_{tx}, \tag{1.3}$$

i.e. the Lagrangian $L = L(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, u_{xt})$ is symmetric in the mixed second partials.

The purpose of this paper is two-fold: (i) to point out that this symmetry assumption is required in the papers by Blakeslee and Logan, and (ii) to state and prove a general result which does not require symmetry in the Lagrangian and which gives conservation laws due to invariance of the variational problem under local, Lie groups of transformations. This result follows from a straightforward proof of the classical Noether theorem (Noether 1918).

2. The general result

For simplicity, we study the variational functional

$$J(u(t^1, t^2)) = \iint_D L(t^1, t^2, u, u_1, u_2, u_{11}, u_{22}, u_{12}, u_{21})\, dt^1\, dt^2, \tag{2.1}$$

where u is four times continuously differentiable in the domain D in t^1t^2-space. Subscripts denote partial differentiation, i.e.

$$u_1 = \partial u/\partial t^1, \qquad u_{12} = \partial^2 u/\partial t^2 \partial t^1, \qquad \text{etc.}$$

Even though $u_{12} = u_{21}$, it is not necessary that $\partial L/\partial u_{12}$ equal $\partial L/\partial u_{21}$. The Euler equation corresponding to (2.1) is

$$E(L) \equiv \frac{\partial L}{\partial u} - \frac{\partial}{\partial t^\alpha} L_{u_\alpha} + \frac{\partial^2}{\partial t^\alpha \partial t^\beta} L_{u_{\alpha\beta}} = 0, \tag{2.2}$$

where $L_{u_\alpha} = \partial L/\partial u_\alpha$, $L_{u_{\alpha\beta}} = \partial L/\partial u_{\alpha\beta}$, with sums being performed over repeated indices. It presents no difficulties to extend (2.1) to several independent variables $t^1, \ldots, t^m$ and/or several dependent functions $u^1, \ldots, u^n$.

We further consider a one-parameter family of transformations

$$\begin{aligned} \bar{t}^\alpha &= t^\alpha + \varepsilon\tau^\alpha(t^1, t^2, u) + o(\varepsilon) \\ \bar{u} &= u + \varepsilon\xi(t^1, t^2, u) + o(\varepsilon), \end{aligned} \tag{2.3}$$

where $\alpha = 1, 2$, ε is a real parameter, and $o(\varepsilon)$ denotes terms which go to zero faster than ε. The functions τ^α and ξ are assumed to be twice continuously differentiable. Again, the analysis can easily be extended to transformations involving several parameters $\varepsilon^1, \ldots, \varepsilon^r$ (see Logan 1977).

Following Logan (1977), we say that $J(u)$ defined by (2.1) is invariant under (2.3) if, and only if,

$$\frac{\partial}{\partial \varepsilon}\left[L\left(\bar{t}^1, \bar{t}^2, \bar{u}, \frac{\partial \bar{u}}{\partial \bar{t}^1}, \frac{\partial \bar{u}}{\partial \bar{t}^2}, \ldots\right) \det\left(\frac{\partial \bar{t}^\alpha}{\partial t^\beta}\right)\right]\Bigg|_{\varepsilon=0} = 0 \tag{2.4}$$

for all functions $u(t^1, t^2)$ which are four times continuously differentiable.

The following theorem is proved by expanding (2.4). It is given in Logan and Blakeslee (1975) and Logan (1977) and does not depend upon symmetry in the Lagrangian.

Theorem. If $J(u)$ defined by (2.1) is invariant under the one-parameter family of transformations (2.3), then

$$\frac{d}{dt^\alpha}(L\tau^\alpha) + L_u C + L_{u_\alpha} \frac{dC}{dt^\alpha} + L_{u_{\alpha\beta}} \frac{d^2 C}{dt^\alpha\, dt^\beta} = 0 \tag{2.5}$$

where

$$C \equiv \xi - u_\alpha \tau^\alpha.$$

Equation (2.5) is an invariance identity relating the Lagrangian L and the transformation generators τ^α and ξ.

We can now obtain a conservation law from (2.5); we assume nothing concerning symmetry in the Lagrangian. First note that

$$L_{u_\alpha}\frac{\mathrm{d}C}{\mathrm{d}t^\alpha}=\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(CL_{u_\alpha})-C\frac{\mathrm{d}}{\mathrm{d}t^\alpha}L_{u_\alpha} \tag{2.6}$$

and

$$\begin{aligned} L_{u_{\alpha\beta}}\frac{\mathrm{d}^2C}{\mathrm{d}t^\alpha\mathrm{d}t^\beta}&=\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left(L_{u_{\alpha\beta}}\frac{\mathrm{d}C}{\mathrm{d}t^\beta}-C\frac{\mathrm{d}}{\mathrm{d}t^\beta}L_{u_{\alpha\beta}}\right)\\ &\quad+C\frac{\mathrm{d}^2}{\mathrm{d}t^\alpha\mathrm{d}t^\beta}L_{u_{\alpha\beta}}+\frac{\mathrm{d}C}{\mathrm{d}t^\beta}\frac{\mathrm{d}}{\mathrm{d}t^\alpha}L_{u_{\beta\alpha}}-\frac{\mathrm{d}C}{\mathrm{d}t^\beta}\frac{\mathrm{d}}{\mathrm{d}t^\alpha}L_{u_{\alpha\beta}}. \end{aligned} \tag{2.7}$$

When (2.6) and (2.7) are substituted into (2.5), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left(L\tau^\alpha+L_{u_\alpha}C+L_{u_{\alpha\beta}}\frac{\mathrm{d}C}{\mathrm{d}t^\beta}-C\frac{\mathrm{d}}{\mathrm{d}t^\beta}L_{u_{\alpha\beta}}\right)+\frac{\mathrm{d}C}{\mathrm{d}t^\beta}\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(L_{u_{\beta\alpha}}-L_{u_{\alpha\beta}})=-CE(L), \tag{2.8}$$

where $E(L)$ is the Euler expression given by (2.2). (It is now easy to see that if L is symmetric in the mixed partials then the second term on the LHS of (2.8) vanishes and the equation by Logan (1977) is obtained.) As (2.8) stands, when $E(L)=0$ we do not obtain an equation in conservation form due to the presence of the second term. However, we note that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left(\frac{\mathrm{d}C}{\mathrm{d}t^\beta}L_{u_{\beta\alpha}}-\frac{\mathrm{d}C}{\mathrm{d}t^\beta}L_{u_{\alpha\beta}}\right)\\ &\quad=\frac{\mathrm{d}C}{\mathrm{d}t^\beta}\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(L_{u_{\beta\alpha}}-L_{u_{\alpha\beta}})+L_{u_{\alpha\beta}}\left(\frac{\mathrm{d}^2C}{\mathrm{d}t^\beta\,\mathrm{d}t^\alpha}-\frac{\mathrm{d}^2C}{\mathrm{d}t^\alpha\,\mathrm{d}t^\beta}\right), \end{aligned} \tag{2.9}$$

and the second term on the RHS of (2.9) vanishes because of the equality of mixed partials since C depends only on τ^α, ξ, and u, all of which possess a high degree of smoothness. Therefore, when $E(L)=0$, (2.8) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left[L\tau^\alpha+\left(L_{u_\alpha}-\frac{\mathrm{d}}{\mathrm{d}t^\beta}L_{u_{\alpha\beta}}\right)C+\frac{\mathrm{d}C}{\mathrm{d}t^\beta}L_{u_{\beta\alpha}}\right]=0, \tag{2.10}$$

which gives the appropriate conservation law due to invariance of (2.1) under (2.3).

3. Examples and remarks

The variational problem (1.1) is obviously invariant under the one-parameter family $\bar{t}=t+\varepsilon$, $\bar{x}=x$, $\bar{u}=u$ of time translations. Here, with $\tau^1=1$, $\tau^2=\xi=0$, it is easy to show that (2.10) gives the conservation law

$$\frac{\partial}{\partial t}(\tfrac{1}{2}u_t^2+\tfrac{1}{2}u_{tx}^2+u_tu_{txx})+\frac{\partial}{\partial x}(\tfrac{1}{2}\delta^2u_t^2-u_{tt}u_{tx})=0.$$

It is interesting to note that the term u_{tx}^2 in the Lagrangian in (1.1) can be partitioned in infinitely many ways giving a class of Lagrangians

$$L=\tfrac{1}{2}\delta^2u_tu_x+\tfrac{1}{2}u_t^2-(a/4)u_{xt}^2-[(2-a)/4]u_{tx}^2, \tag{3.1}$$

where a is a real parameter, all of which lead to the same Euler equation. But now infinitely many conservation laws result, one for each a. They are

$$(\partial/\partial t)\{\tfrac{1}{2}u_t^2+\tfrac{1}{2}(1-a)u_{xt}^2+[2(2-a)/4]u_{txx}u_t\}$$
$$+(\partial/\partial x)\{\tfrac{1}{2}\delta^2u_t^2+(a/2)u_{xtt}u_t-[(2-a)/2]u_{tx}u_{tt}\}=0. \quad (3.2)$$

Choosing different values of a has the effect of shifting terms back and forth from the 'density' term to the 'flux' term in (3.2). In the second-order case, therefore, there is not a unique Lagrangian or conservation law associated with a single governing equation of motion. Physical arguments may be required to select an appropriate value of the parameter a in (3.1).

References

Blakeslee J S and Logan J D 1976 *Nuovo Cimento* **34** B 319-24
—— 1977 *J. Phys. A: Math. Gen.* **10** 1353-9
Logan J D 1977 *Invariant Variational Principles* (New York: Academic) pp 121-4
Logan J D and Blakeslee J S 1975 *J. Math. Phys.* **16** 1374-1379
Noether E 1918 *Nachr. Akad. Wiss. Gottingen, Math.-Phys. Kl* **II** 235-57